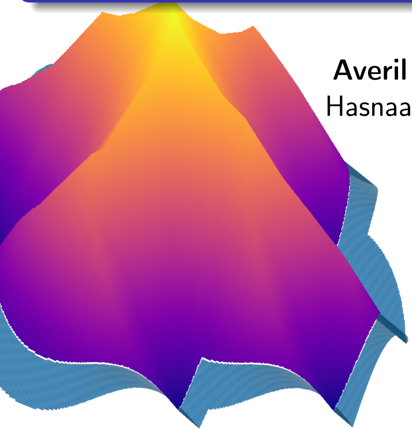
D_μ vs $\langle \cdot, \cdot \rangle_\mu^\pm$
Equivalence between two notions of viscosity solutions in the Wasserstein space



Averil Prost (LMI, INSA Rouen Normandie)
Hasnaa Zidani (LMI, INSA Rouen Normandie)

March 22, 2024
ANR COSS Meeting

Aim of the talk

We consider a first-order Hamilton-Jacobi equation of the form

$$H\left(\mu, D_\mu V(\mu)\right) = 0 \quad \mu \in \Omega, \qquad V(\mu) = \mathfrak{J}(\mu) \quad \mu \in \partial\Omega. \tag{1}$$

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Our aim Compare two notions of viscosity solutions for (1).

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In $\mathbb{R}^d$, viscosity solutions of $H(x, \nabla_x u) = 0$ are equivalently defined using

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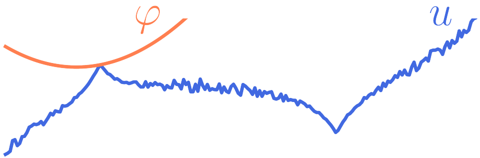

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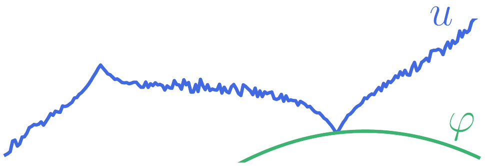

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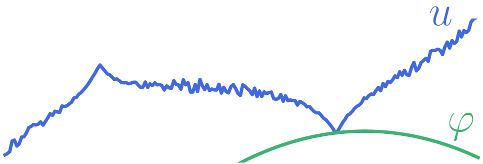

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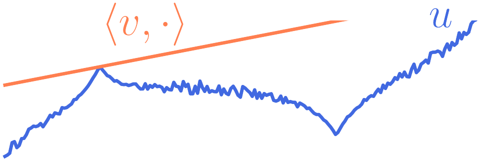

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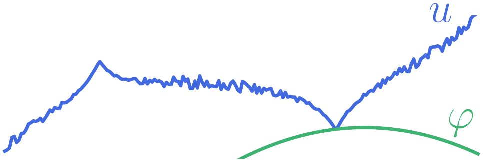

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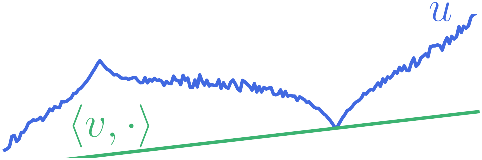

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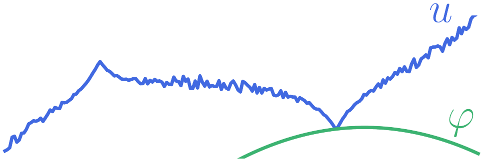

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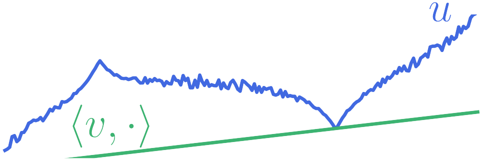

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Both are linked by $\nabla\varphi(x) = v$. Extension to viscosity in measure spaces?

Notations

Let μ, ν be two probability measures on $\mathbb{R}^d$. If $f : \mathbb{R}^d \to Y$ is measurable, the pushforward $f\#\mu$ is a measure on Y given by $(f\#\mu)(A) = \mu(f^{-1}(A))$ for any measurable $A \subset Y$.

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$$\Gamma(\mu, \nu) := \left\{ \eta = \eta(x, y) \in \mathscr{P}_2(\mathbb{R}^d \times \mathbb{R}^d) \;\middle|\; \pi_x\#\eta = \mu, \;\; \pi_y\#\eta = \nu \right\}.$$

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Def – Wasserstein space The Wasserstein space $\mathscr{P}_2(\mathbb{R}^d)$ is the set of measures μ such that $d^2_{\mathcal{W}}(\mu,\delta_0)$ is finite, endowed with the Wasserstein distance.

Viscosity solutions in $\mathscr{P}_2(\mathbb{R}^d)$

Using **Lions differentiability**, introduced in [Lio07].

- Represent any $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ as the law of a set of random variables $X \in L^2_{\mathbb{P}}(E, \mathcal{E}; \mathbb{R}^d)$, that is, $\mu = X\#\mathbb{P}$. Then any function $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ can be *lifted* in

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- Provides a definition of $\mathcal{C}^1$ functions and higher derivatives [Sal23], used to obtain existence of "regular" solutions to mean-field games [CDLL19, CP20, MZ22] and viscosity solutions [PW18, BY19, DJS23]...

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Using **semidifferentials** in a well-chosen tangent space [AGS05]. Usually taken as

$$\mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) := \overline{\{\nabla\varphi \mid \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)\}}^{L^2_\mu(\mathbb{R}^d;\mathsf{T}\mathbb{R}^d)}.$$

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Other ideas: applying metric viscosity [AF14, GŚ15], linear derivatives [FN12, BIRS19], pathwise solutions [WZ20, CGK$^+$23], directional derivatives [Jer22, JPZ23].

Using directional derivatives

Idea: define the Hamiltonian over a set of functions, as

$$H : \mathbb{T} \to \mathbb{R},$$

where $\mathbb{T}$ is a set of pairs (μ, p) with $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ and $p : \mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$.

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For instance,

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- ❓ **Is it possible to reformulate using semidifferentials?** ❓

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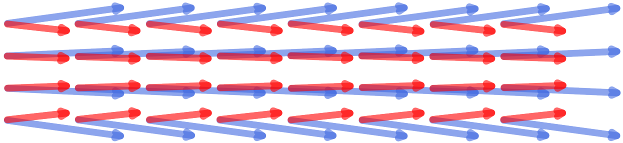

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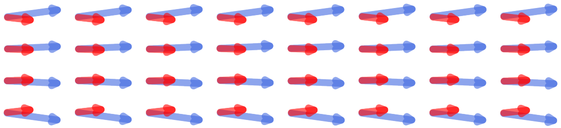

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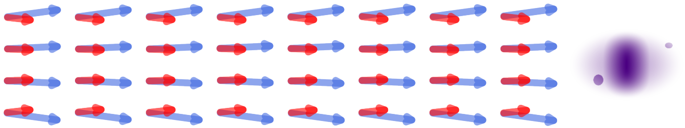

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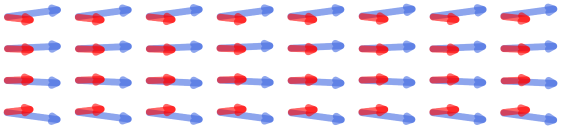

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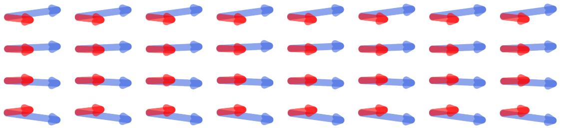

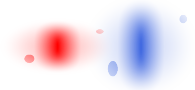

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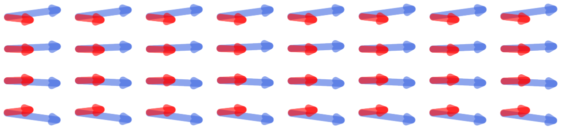

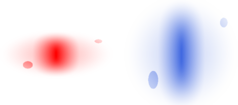

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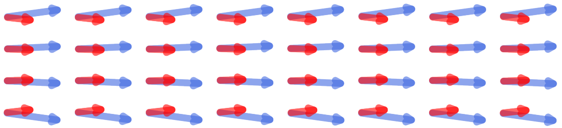

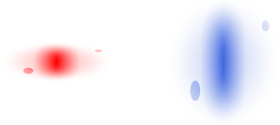

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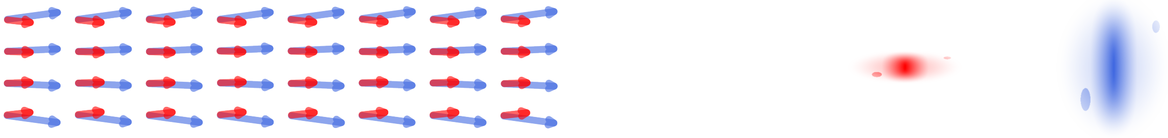

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Generalized tangent space

Let $\mu, \nu \in \mathscr{P}_2(\mathbb{R}^d)$, and denote

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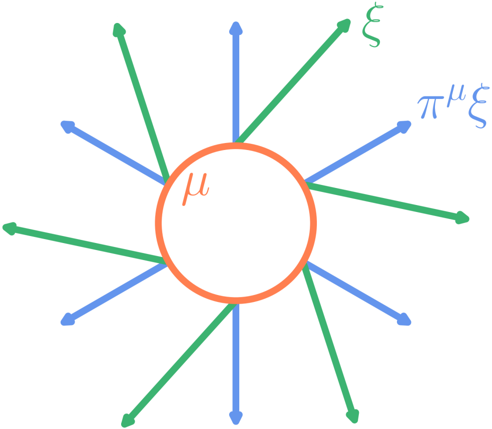

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$$\text{Bary}\,(\xi)(x) = \int_{v \in \mathsf{T}_x\mathbb{R}^d} v d\xi_x(v).$$

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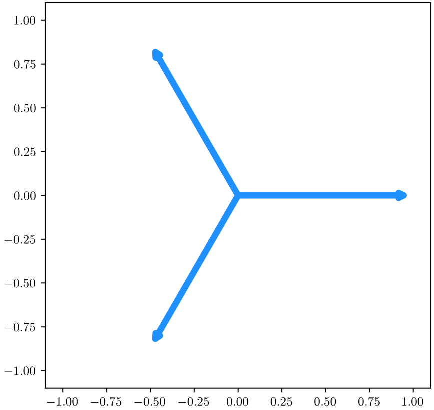

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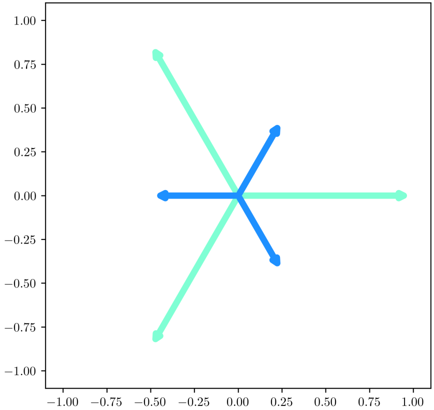

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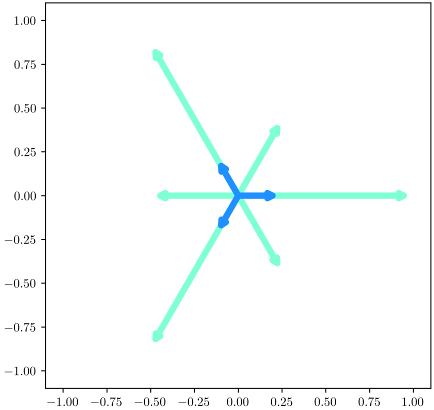

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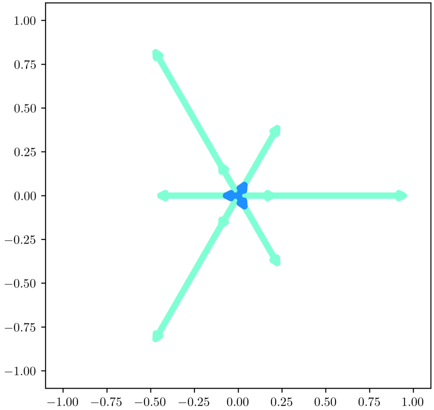

Directional derivatives

For an application $\varphi : \mathscr{P}_2(\mathrm{T}\mathbb{R}^d) \to \mathbb{R}$, we denote

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Def – Metric cotangent bundle Let

$$\mathbb{T}_\mu := \left\{ p : \textbf{Tan}_\mu\mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R} \;\middle|\; \begin{array}{l} p \text{ is positively homogeneous and} \\ \text{Lipschitz-continuous w.r.t. } W_\mu. \end{array} \right\}$$

Denote $\mathbb{T} := \bigcup_{\mu\in\mathscr{P}_2(\mathbb{R}^d)} \{\mu\} \times \mathbb{T}_\mu$.

Precise definition of the Hamiltonian

The Hamiltonian is defined as an application

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- Give a notion of viscosity solutions using the semidifferentials of [AF14].
- Compare it with a notion of viscosity solutions using test functions.

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Pseudo scalar products

Denote $0_\mu := (\pi_x, 0)\#\mu$ and $\|\xi\|_\mu := W_\mu(\xi, 0_\mu)$.

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Expanding the definition of W_μ yields

$$\langle \xi, \zeta \rangle_\mu^+ = \sup_{\eta \in \Gamma_\mu(\xi,\zeta)} \int_{(x,v,w)} \langle v, w \rangle \, d\eta, \qquad \text{and} \qquad \langle \xi, \zeta \rangle_\mu^- = \inf_{\eta \in \Gamma_\mu(\xi,\zeta)} \int_{(x,v,w)} \langle v, w \rangle \, d\eta.$$

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Properties of $\langle \cdot, \cdot \rangle_\mu^\pm$

If $\xi = f\#\mu$ and $\zeta = g\#\mu$ for some $f, g \in L^2_\mu(\mathbb{R}^d; \mathsf{T}\mathbb{R}^d)$, then

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For instance, if $\xi = \frac{1}{2}\delta_{(0,v_0)} + \frac{1}{2}\delta_{(0,v_1)}$, then $\beta := \frac{1}{2}\delta_{(0,v_0,v_1)} + \frac{1}{2}\delta_{(0,v_1,v_0)}$ yields

$$\langle \xi, \xi \rangle_\mu^- \leqslant \frac{1}{2}\langle v_0, v_1 \rangle + \frac{1}{2}\langle v_1, v_0 \rangle = -1 = -\|\xi\|_\mu^2.$$

Convexity properties

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Proposition 1 Let ξ_0, $\xi_1 \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ and $\beta \in \Gamma_\mu(\xi_0, \xi_1)$. Then for any $\zeta \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$,

$$[0,1] : t \mapsto \left\langle \zeta, \xi_t^\beta \right\rangle_\mu^+ \quad \text{is convex,} \qquad [0,1] : t \mapsto \left\langle \zeta, \xi_t^\beta \right\rangle_\mu^- \quad \text{is concave.}$$

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Let $A, B \subset \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ be nonempty, horizontally convex and bounded sets, with A compact w.r.t. the topology induced by W_μ. Then

$$\sup_{\alpha \in A} \inf_{\beta \in B} \langle \alpha, \beta \rangle_\mu^\pm = \inf_{\beta \in B} \sup_{\alpha \in A} \langle \alpha, \beta \rangle_\mu^\pm .$$

Fréchet sub and superdifferentials [AF14, Definition 4.7]

Def – Superdifferential Let $\varphi : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$. An element $\xi \in \mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$ belongs to the superdifferential of φ at μ, denoted $\partial_\mu^+ \varphi$, if for all $\nu \in \mathscr{P}_2(\mathbb{R}^d)$,

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Example

Given $A \subset \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)_\mu$, let again $\overline{\text{conv}}A$ be the smallest closed set B containing A such that

$$\forall \xi_0, \xi_1 \in B, \qquad \forall \beta \in \Gamma_\mu(\xi_0, \xi_1), \qquad \forall t \in [0,1], \qquad \xi_t^\beta = (\pi_x, (1-t)\pi_v + t\pi_w)\#\beta \in B.$$

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Proposition 2 Let $\varphi : \mu \mapsto d_{\mathcal{W}}^2(\mu, \sigma)$ for some fixed $\sigma \in \mathscr{P}_2(\mathbb{R}^d)$. The superdifferential of φ is everywhere nonempty and given by

$$\partial_\mu^+ \varphi = \overline{\text{conv}} \left\{ -2 \cdot \xi \mid \xi \in \overrightarrow{\mu\sigma} \right\}.$$

For reference, the gradient of $x \mapsto |x-y|^2$ at x is $2(x-y) = -2(y-x)$.

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Test function spaces

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$$\mathscr{T}_{+,\mu} := \left\{ \varphi : \Omega \to \mathbb{R} \;\middle|\; \begin{array}{c} \varphi \text{ is lower semicontinuous, directionally differentiable at } \mu, \\ \partial_\mu^+ \varphi \text{ is nonempy, bounded and } D_\mu \varphi(\mu)(\cdot) = \inf_{\zeta \in \partial_\mu^+ \varphi} \langle \cdot, \zeta \rangle_\mu^- . \end{array} \right\}.$$

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- Does not appeal to the theory of Wasserstein gradient.
- Retains a link between directional derivatives and semidifferentials.

Examples of test functions

Proposition 3 Let $\sigma \in \mathscr{P}_2(\mathbb{R}^d)$ be fixed. Then the function $d^2_{\mathcal{W}}(\cdot, \sigma)$ belongs to $\mathscr{T}_{+,\mu}$ for any $\mu \in \mathscr{P}_2(\mathbb{R}^d)$.

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From [Gig08, Proposition 4.10], there holds that

$$D_\mu d_{\mathcal{W}}^2(\cdot, \sigma)(\xi) = \inf_{\eta \in -2 \cdot \overrightarrow{\mu\sigma}} \langle \xi, \eta \rangle_\mu^- .$$

Examples of test functions

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From [Gig08, Proposition 4.10], there holds that

$$D_\mu d^2_{\mathcal{W}}(\cdot, \sigma)(\xi) = \inf_{\eta \in -2 \cdot \overrightarrow{\mu\sigma}} \langle \xi, \eta \rangle^-_\mu .$$

Proposition 4 Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ and $\zeta \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ be fixed. Then the function $\varphi : \nu \mapsto \inf_{\eta \in \overrightarrow{\mu\nu}} \langle \eta, \zeta \rangle^-_\mu$ belongs to $\mathscr{T}_{+,\mu}$, and there holds

$$D_\mu \varphi(\xi) = \langle \xi, \zeta \rangle^-_\mu .$$

The notion of viscosity

Consider the HJB equation

$$H(\mu, D_\mu u(\mu)) = 0 \quad \mu \in \Omega, \qquad u(\mu) = \mathfrak{J}(\mu) \quad \mu \in \partial\Omega. \tag{2}$$

Def – Using test functions
A map $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is a subsolution of (2) if it is **u.s.c**, if $u \leqslant \mathfrak{J}$ over $\partial\Omega$, and if for any μ and $\varphi \in \mathscr{T}_{+,\mu}$ such that $u - \varphi$ reaches a **maximum** at μ,

$$H(\mu, D_\mu \varphi) \leqslant 0.$$

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Table of Contents

Statement

Assume that $H : \mathbb{T} \to \mathbb{R}$ satisfies

$$\begin{aligned} \forall \varphi \in \mathscr{T}_{+,\mu}, \qquad & H\left(\mu, D_\mu \varphi\right) \leqslant \sup_{\xi \in \partial_\mu^+ \varphi} H\left(\mu, \langle \xi, \cdot \rangle_\mu^-\right), \\ \forall \varphi \in \mathscr{T}_{-,\mu}, \qquad & H\left(\mu, D_\mu \varphi\right) \geqslant \inf_{\xi \in \partial_\mu^- \varphi} H\left(\mu, \langle \xi, \cdot \rangle_\mu^+\right). \end{aligned} \tag{hyp-H}$$

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Theorem Assume that (hyp-H) is satisfied. Then a map $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is a viscosity subsolution (resp. supersolution) in the sense of test functions if and only if it is a viscosity subsolution (resp. supersolution) in the sense of semidifferentials.

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- Given an element $\zeta \in \partial^+_\mu u$, build a test function φ such that $D_\mu \varphi(\xi) = \langle \xi, \zeta \rangle^-_\mu$.
- Given a test function, use the representation of $D_\mu \varphi$ and (hyp-H).

Examples of applications

- **Eikonal-type Hamiltonians** Let $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ be nondecreasing.

$$H : \mathbb{T} \to \mathbb{R}, \qquad H(\mu, p) := \sup_{\xi \in \textbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d),\ \|\xi\|_\mu = 1} \kappa\left(|p(\xi)|\right).$$

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- **"Concave-convex" Hamiltonians** Let F_1 and $F_2 : \mathscr{P}_2(\mathbb{R}^d) \rightrightarrows \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)$ be set-valued maps such that for any $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ and $i \in \{1, 2\}$, $F_i[\mu]$ is a nonempty, horizontally convex and compact subset of $\mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$ endowed with W_μ.

$$H : \mathbb{T} \to \mathbb{R}, \qquad H(\mu, p) := \sup_{\xi_1 \in F_1[\mu]} -p(\xi_1) + \inf_{\xi_2 \in F_2[\mu]} -p(\xi_2).$$

Conclusion and perspectives

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- Extension over $\mathscr{P}_2(\mathcal{N})$, where $\mathcal{N}$ is not Hilbertian (network structure).
- Link with Lions differentiability?

Thank you!

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