# Tropical heat

## The eikonal equation as a (max,+) version of the Poisson equation

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January 30, 2024
LMI-LMRS Doctoral seminar

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# The (max,+) idempotent calculus

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Then $(\Omega, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ is a semiring (ring without additive inverse).

# Integrals

Endow $\Omega$ with the application $\mathbb{d}(a,b) := \left|e^a - e^b\right|$, and let $f : \Omega \to \Omega$ be continuous. Then

$$\sum_{i\in\mathbb{Z}\cup\{-\infty\}}^{\oplus} f(hi) = \max_{i\in\mathbb{Z}\cup\{-\infty\}} f(hi) \xrightarrow[h\searrow 0]{} \sup_{x\in\Omega=\mathbb{R}\cup\{-\infty\}} f(x).$$

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In particular, the scalar product becomes

$$\langle f,g\rangle_{\oplus} = \int_{x\in\Omega}^{\oplus} f(x)\otimes g(x) = \sup_{x\in\Omega} f(x)+g(x).$$

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> **Motivation** Ease the study of **optimization problems** by directly working with the "natural" operations of (here) **maximization and sum** of gains.

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# An elementary result

**Proposition –** ✎ Let $f > g$. Then

$$\lim_{h \searrow 0} h \log \left( e^{f/h} + e^{g/h} \right) = \lim_{h \searrow 0} h \log \left( e^{f/h} - e^{g/h} \right) = f.$$

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Going further, for any upper bounded and continuous $f$, there holds

$$\lim_{h \searrow 0} h \log\left(\int_{x \in \mathbb{R}^d} \exp\left(\frac{f(x)}{h}\right) dx\right) = \sup_{x \in \mathbb{R}^d} f(x).$$

# Logarithm transform

**Def 3 – Logarithm trick** For $h > 0$, consider the operations

$$a \otimes_h b := h \log\left(e^{a/h} \cdot e^{b/h}\right), \qquad a \oplus_h b := h \log(e^{a/h} + e^{b/h}).$$

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Then

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$$a \hat{}^{\oplus} b := \lim_{h \searrow 0} h \log\left(\exp(a/h) \hat{}\, \exp(b/h)\right) = \begin{cases} a & b = 0 \\ \mathbb{1} = 0 & b < 0 \\ +\infty & b > 0. \end{cases}$$

# Dialog between $(+, \cdot)$ and $(\max, +)$

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Classical operations

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| --- |
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$h\log(\cdot)$ $\rightharpoonup$

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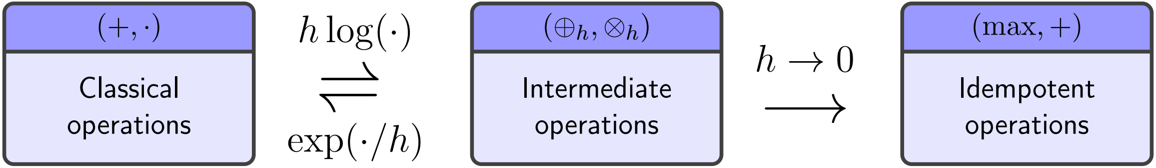

# Linearity

Let $\mathrm{I} : \mathbb{R}^d \to \mathbb{R}^d$ be linear, and denote

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The limit operator $\Phi := \lim_{h \searrow 0} \Phi_h$ is $(\max, +)-$linear.

# Example

Let $d = 2$ and $I(x) = Ax$, where $A := \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$. Then

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# Setting

Consider the $(\max, +)$ system

$$\Pi \otimes \xi = \beta, \tag{1}$$

where $\Pi \in \mathbb{M}_{2,2}$ is a matrix, $\xi, \beta \in \mathbb{R}^2$ are vectors with $\beta$ given, and for each $i \in [\![1, 2]\!]$,

$$(\Pi \otimes \xi)_i = \sum_{j \in [\![1,2]\!]}^{\oplus} \Pi_{ij} \otimes \xi_j = \max_{j \in [\![1,2]\!]} \Pi_{ij} + \xi_j .$$

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In which cases can we get a solution to (1) by using the link with classical algebra?

# Transformation into linear variables

For each $h > 0$, define $P \in \mathbb{M}_{2,2}$ and $x, b \in \mathbb{R}^2$ by

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The linear system $P^h x^h = b^h$ has a solution if $\det(P^h) \neq 0$, i.e.

$$P_{11}^h P_{22}^h - P_{12}^h P_{21}^h \neq 0 \qquad \iff \qquad \exp\left(\frac{\Pi_{11} + \Pi_{22}}{h}\right) \neq \exp\left(\frac{\Pi_{12} + \Pi_{21}}{h}\right).$$

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Assume that $\Pi_{11} + \Pi_{22} \neq \Pi_{12} + \Pi_{21}$, and consider $x^h = (P^h)^{-1} b^h$ solving $P^h x^h = b^h$.

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The linear system $P^h x^h = b^h$ has a solution if $\det(P^h) \neq 0$, i.e.

$$P_{11}^h P_{22}^h - P_{12}^h P_{21}^h \neq 0 \qquad \iff \qquad \exp\left(\frac{\Pi_{11} + \Pi_{22}}{h}\right) \neq \exp\left(\frac{\Pi_{12} + \Pi_{21}}{h}\right).$$

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Then $P^{-1} = \mathbb{I}_d$. As (2) is satisfied for all $b = \exp(\beta/h)$, we obtain that $\xi = \beta$ solves (1).

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Since $\det P^h = e^{4/h} - e^{6/h} < 0$, the condition (2) over $\beta$ becomes

$$e^{\frac{3+\beta_1}{h}} - e^{\frac{4+\beta_2}{h}} \leqslant 0, \quad \text{and} \quad -e^{\frac{2+\beta_1}{h}} + e^{\frac{1+\beta_2}{h}} \leqslant 0.$$

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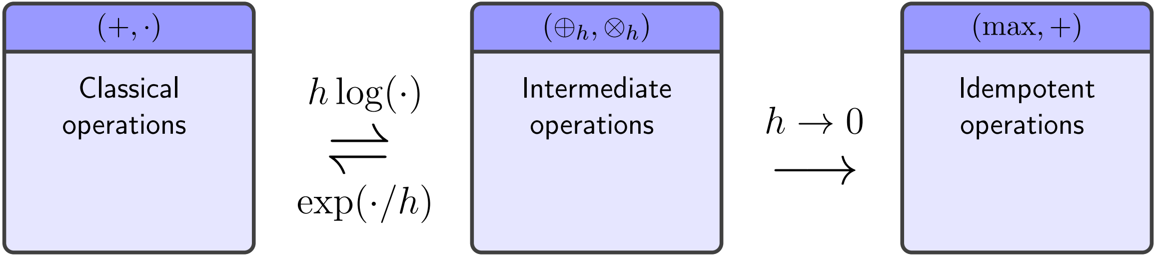

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| $(+, \cdot)$ | | $(\oplus_h, \otimes_h)$ | | $(\max, +)$ |
|---|---|---|---|---|
| Classical operations<br>Linear systems | $h \log(\cdot)$ $\rightleftharpoons$ $\exp(\cdot/h)$ | Intermediate operations<br>Intermediate values | $h \to 0$<br>$\longrightarrow$ | Idempotent operations<br>$(\max, +)$ systems |

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so that after dividing by $\frac{\exp(v/h)}{h} > 0$, we get again $\partial_t v(t,x) - \langle \nabla v(t,x), b(t,x) \rangle = 0$.

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or after simplification,

$$\partial_t v(t,x) - h\Delta v(t,x) - \left|\nabla v(t,x)\right|^2 = 0.$$

# A link with viscosity solutions

**Def 4 – (Historical intuition, see [CL83])** The *vanishing viscosity* solution of

$$\partial_t v - |\nabla v|^2 = 0$$

is the limit when $h$ goes to 0 of the (unique) solution of the equation

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It may be characterized by two sign inequalities that maintain the validity of the comparison principle coming from the elliptic perturbation.

# The heat kernel

**Def 5 – Heat kernel** Let $C_d$ be a normalizing constant, and for any $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$, define

$$\mu_{t,x}^h := \frac{1}{\sqrt{S_d th}} \exp\left(-\frac{|\cdot - x|^2}{2th}\right) \mathcal{L}_{\mathbb{R}^d} \in \mathscr{P}_2(\mathbb{R}^d).$$

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In particular, $\mu_{t,x}^h \underset{t \searrow 0}{\longrightarrow} \delta_x$ narrowly and in the Wasserstein topology.

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In particular, $\mu^h_{t,x} \underset{t \searrow 0}{\longrightarrow} \delta_x$ narrowly and in the Wasserstein topology.

**Proposition – Kernel representation** The (weak) solution of the heat equation $\partial_t u(t,x) - h\Delta u(t,x) = 0$ with initial value $u_0$ is given by

$$u(t,x) = \int_{y \in \mathbb{R}^d} u_0(y) d\mu^h_{t,x}(y).$$

# Going through the transformation

Let $u_0 > 0$ upper bounded and continuous.

# Going through the transformation

Let $u_0 > 0$ upper bounded and continuous. Denoting again $v = h \log(u)$, and $v_0 = h \log(u_0)$,

$$v(t,x) = h \log \left( \int_{y \in \mathbb{R}^d} \exp\left( \frac{v_0(y)}{h} \right) d\mu^h_{t,x}(y) \right)$$

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&= h \log \left( \frac{1}{\sqrt{C_d t h}} \int_{y \in \mathbb{R}^d} \exp\left(\frac{v_0(y)}{h}\right) \exp\left(-\frac{|y-x|^2}{2th}\right) dy \right)
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$$= -\frac{h}{2}\log(C_d t h) + h\log\left(\int_{y\in\mathbb{R}^d} \exp\left(\frac{1}{h}\left[v_0(y) - \frac{|y-x|^2}{2t}\right]\right) dy\right)$$

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&= -\frac{h}{2}\log\left(C_d t h\right) + h\log\left(\int_{y\in\mathbb{R}^d} \exp\left(\frac{1}{h}\left[v_0(y) - \frac{|y-x|^2}{2t}\right]\right) dy\right) \\
&\underset{h\searrow 0}{\longrightarrow} 0 + \sup_{y\in\mathbb{R}^d}\left[v_0(y) - \frac{|y-x|^2}{2t}\right].
\end{aligned}$$

# Control interpretation

Let

$$\hat{m}_{t,x} := -\frac{|\cdot - x|^2}{2t}, \qquad \hat{\mu}_{t,x}(B) := \sup_{y \in B} \hat{m}_{t,x}(y) \quad \forall B \subset \mathbb{R}^d.$$

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**Proposition – Value function [Lio82]** The function

$$V(t,x) := \int_{y\in\mathbb{R}^d}^{\oplus} v_0 \otimes \hat{\mu}_{t,x} = \sup_{y\in\mathbb{R}^d} \left[ v_0(y) - \frac{|y-x|^2}{2t} \right] \tag{4}$$

is the unique viscosity solution of the Hamilton-Jacobi equation $\partial_t V - |\nabla V|^2 = 0$ such that $V(0,\cdot) = v_0$.

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In the vocabulary of [DMD99], $\hat{m}_{t,x}$ is the *density* of the *Maslov measure* $\hat{\mu}_{t,x}$.

**Proposition – Value function [Lio82]** The function

$$V(t,x) := \int_{y \in \mathbb{R}^d}^{\oplus} v_0 \otimes \hat{\mu}_{t,x} = \sup_{y \in \mathbb{R}^d} \left[ v_0(y) - \frac{|y-x|^2}{2t} \right] \tag{4}$$

is the unique viscosity solution of the Hamilton-Jacobi equation $\partial_t V - |\nabla V|^2 = 0$ such that $V(0,\cdot) = v_0$. The formula (4) is known as the Hopf-Lax formula.

# Going further

- In the case of the heat equation, **Hopf-Cole** transform. But Lax-Hopf formula valid for a larger class of HJ equations of the type

$$\partial_t V(t,x) + H\left(x, \nabla V(t,x)\right) = 0,$$

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- Maslov measures may be used to recast the Lax-Hopf semigroup as the conditional expectation of Maslov stochastic processes.
- Using the Hopf-Lax semigroup, Maslov defined weak solution by "duality", in the spirit of

$$\langle u, \varphi \rangle_{\oplus} = \langle u_0, S_t^* \varphi \rangle_{\oplus},$$

where $S_t^*$ is a "dual" semigroup acting on test functions $\varphi$ [KM97, Definition 3.1].

# Conclusion

| $(+, \cdot)$ | $(\oplus_h, \otimes_h)$ | $(\max, +)$ |
|---|---|---|
| Classical operations | Intermediate operations | Idempotent operations |
| Linear systems | Intermediate values | $(\max, +)$ systems |
| Heat equation | Viscous Eikonal equation | Eikonal equation |

$h \log(\cdot)$ $\rightleftharpoons$ $\exp(\cdot/h)$ (between $(+,\cdot)$ and $(\oplus_h, \otimes_h)$)

Hopf-Cole $\longrightarrow$ (from Heat equation to Viscous Eikonal equation)

$h \to 0$ $\longrightarrow$ (from $(\oplus_h, \otimes_h)$ to $(\max, +)$)

## Thank you!

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