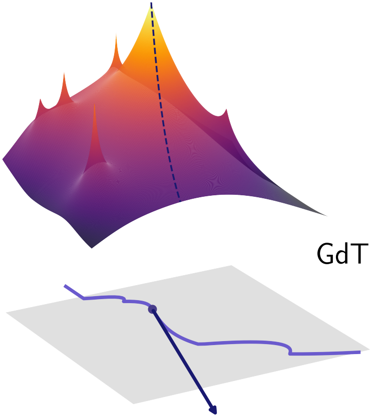
Quadratic is the new smooth

A notion of viscosity for control problems in the Wasserstein space over $\mathbb{R}^d$



Averil Prost

April 11, 2023

GdT Optimisation et contrôle

INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES ROUEN

anr

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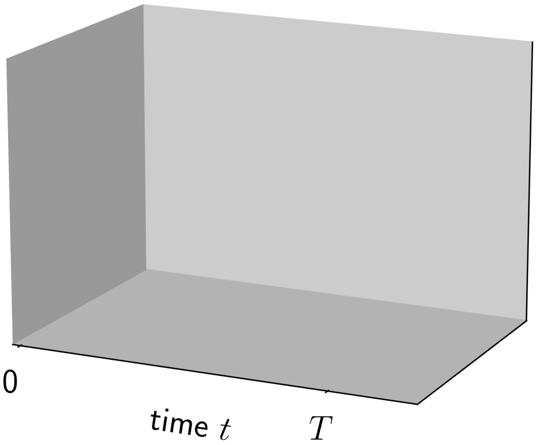

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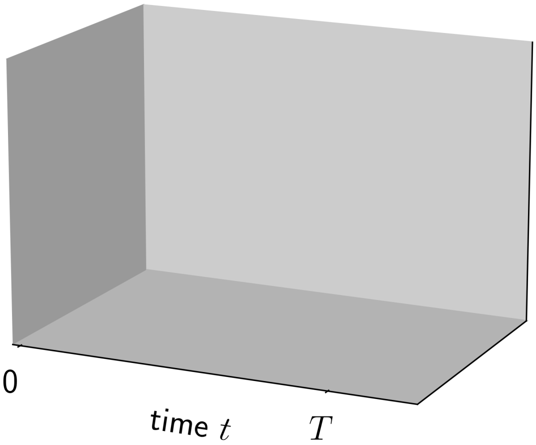

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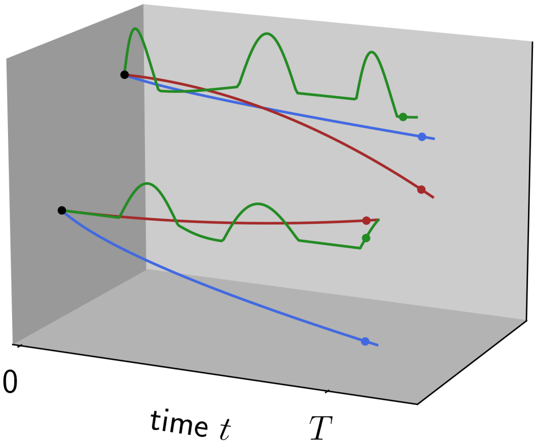

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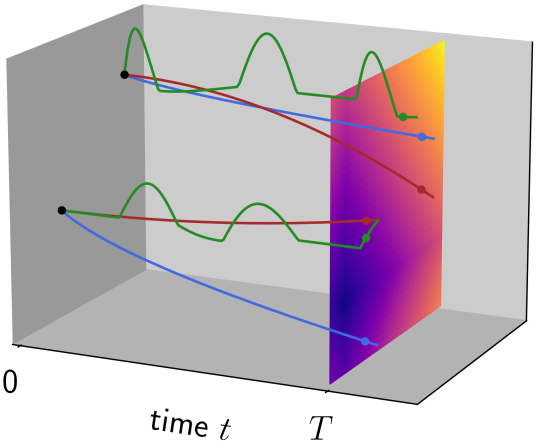

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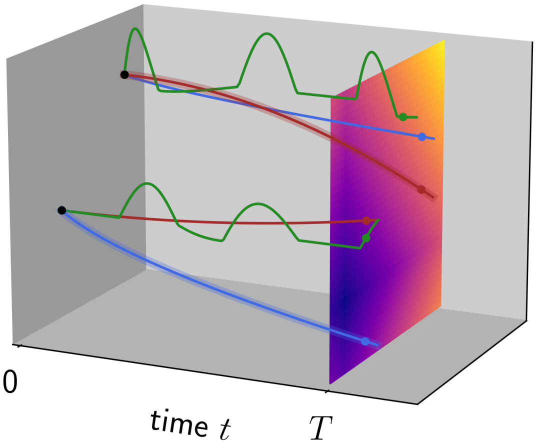
Given $x \in \Omega$, find $u(\cdot)$ such that

$$\mathcal{J}(y_T^{0,x,u}) \leqslant \mathcal{J}(y_T^{0,x,v}) \quad \forall v \in \mathcal{U}_{[0,T]}.$$



Intuition of the Hamilton-Jacobi approach

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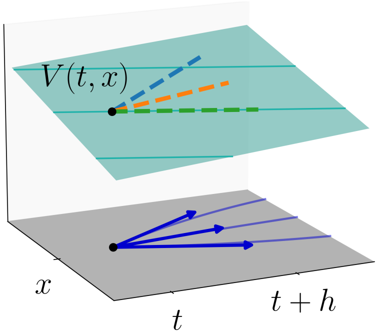

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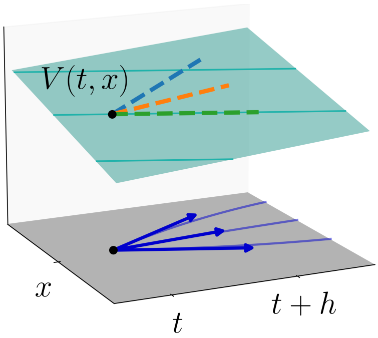

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Another definition by super/subdifferentials.

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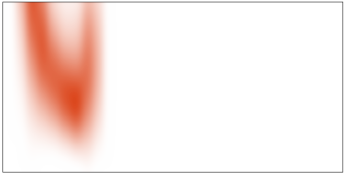

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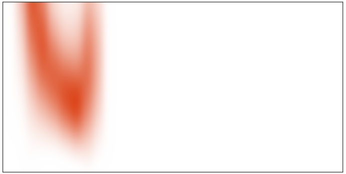

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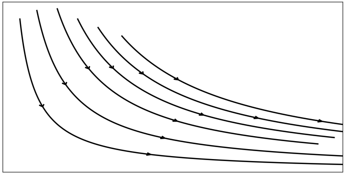

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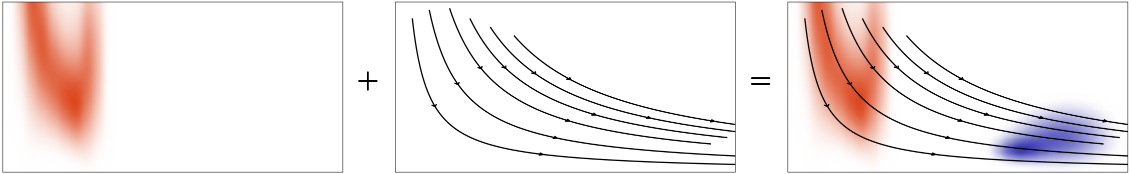

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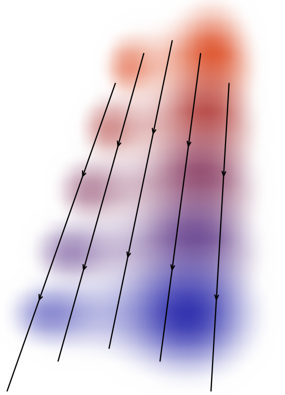

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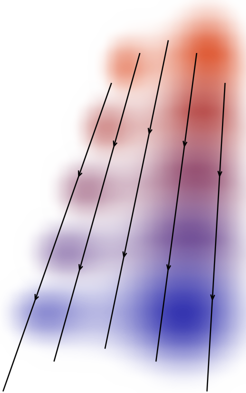

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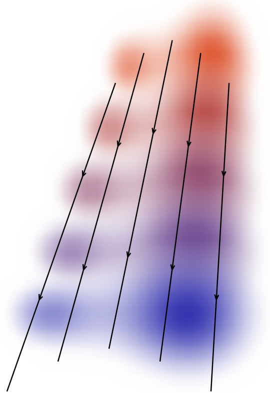

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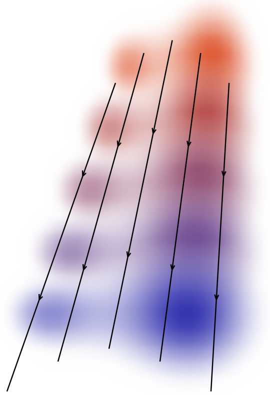
- $\pi^\mu : \mathscr{P}_\mu(T\mathbb{R}^d) \mapsto T_\mu\mathscr{P}_2(T\mathbb{R}^d)$ a partially defined projection.



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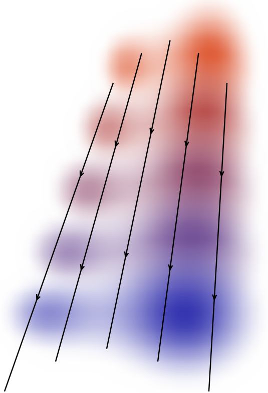
$$\mu_t := ((1-t)\pi_x + t\pi_y)\#\eta = (\pi_x + t(\pi_y - \pi_x))\#\eta = (\pi_x + t\pi_v)\#[(\pi_x, \pi_y - \pi_x)\#\eta].$$

Define $\mathscr{P}_\mu(T\mathbb{R}^d)$ as the set of *initial velocities* $\xi \in \mathscr{P}(T\mathbb{R}^d)$ such that $\pi_x\#\xi = \mu$, and

- $\mathscr{P}_{\mu,o}(T\mathbb{R}^d) := \left\{(\pi_x, \pi_y - \pi_x)\#\eta \mid \eta \in \Gamma_o\left(\mu, \mathscr{P}_2(\mathbb{R}^d)\right)\right\}$,
- $\exp_\mu(t \cdot \xi) := (\pi_x + t\pi_v)\#\xi \qquad \forall \xi \in \mathscr{P}_\mu(T\mathbb{R}^d)$,
- $W_\mu\left(\xi, \overline{\xi}\right) := \lim_{t \searrow 0} \frac{d_{\mathcal{W}}\left(\exp_\mu(t\cdot\xi), \exp_\mu(t\cdot\overline{\xi})\right)}{t}$,
- $T_\mu\mathscr{P}_2(\mathbb{R}^d) := \overline{\mathscr{P}_{\mu,o}(T\mathbb{R}^d)}^{W_\mu}$,
- $\pi^\mu : \mathscr{P}_\mu(T\mathbb{R}^d) \mapsto T_\mu\mathscr{P}_2(T\mathbb{R}^d)$ a partially defined projection.

Now $t \mapsto \exp_\mu(t \cdot \xi)$ is a measure analogue of $t \mapsto x + tv$.



Moving (2/2): the continuity equation

We follow solutions $(\mu_s^{t,\nu,u})_{s\in[t,T]}$ of the controlled nonlocal continuity equation (see [AGS05])

$$\mu_t = \nu, \qquad \partial_s \mu_s + \operatorname{div}\left(f(\cdot, \mu_s, u(s))\mu_s\right) = 0. \tag{CE}$$

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Solution of (CE) Assume that $f : \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d) \times U \mapsto T\mathbb{R}^d$ is Lip. and bounded.

$$\mu_s = S_s^{t,\nu,u}\#\nu, \qquad \text{where } S_t^{t,\nu,u}(x) = x \text{ and } \partial_t S_s^{t,\nu,u} = f\left(S_s^{t,\nu,u},\mu_s,u(s)\right).$$

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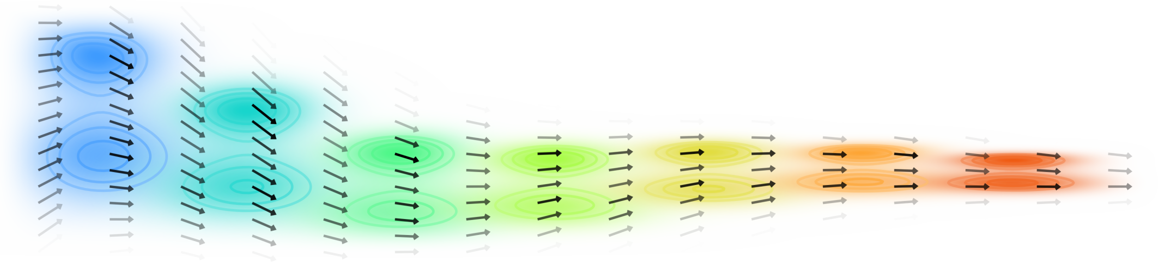

Issue 1: Non-smoothness of the distance

Two conflicting notions of *straight lines*:

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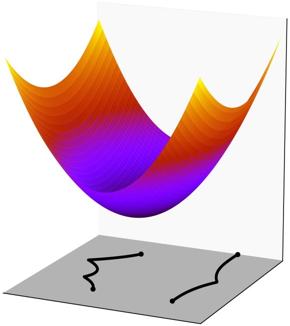

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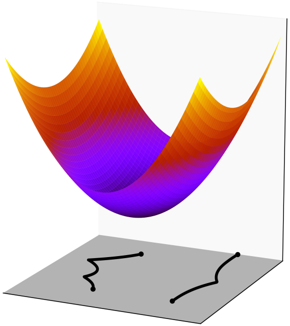
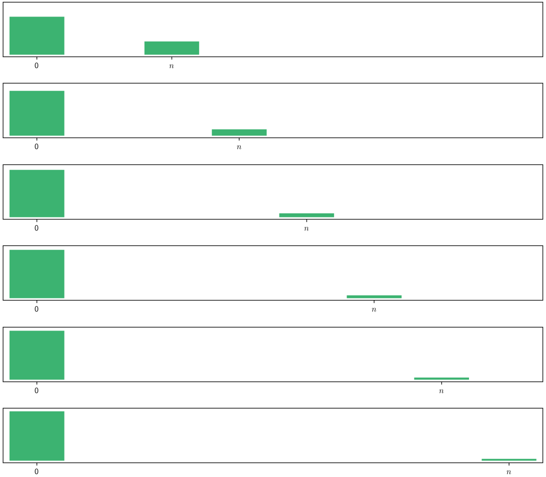
Hence directionally differentiable along $t \mapsto \exp_\mu(t\cdot\xi)$!



Issue 2: lack of local compactness



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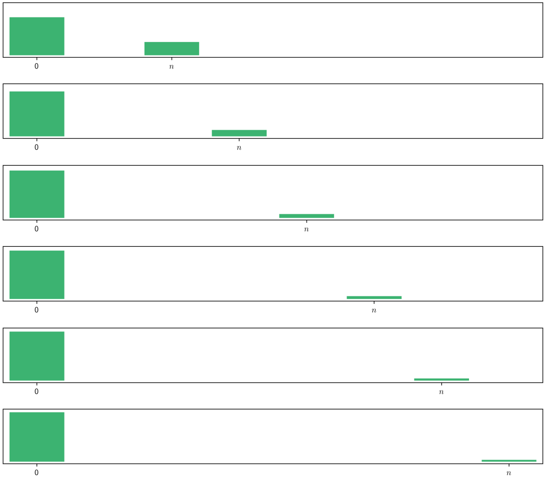

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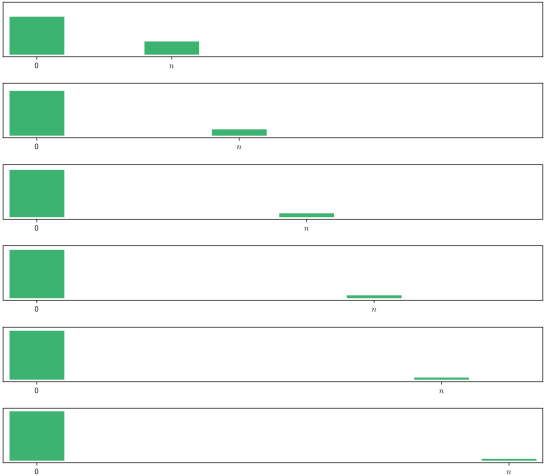

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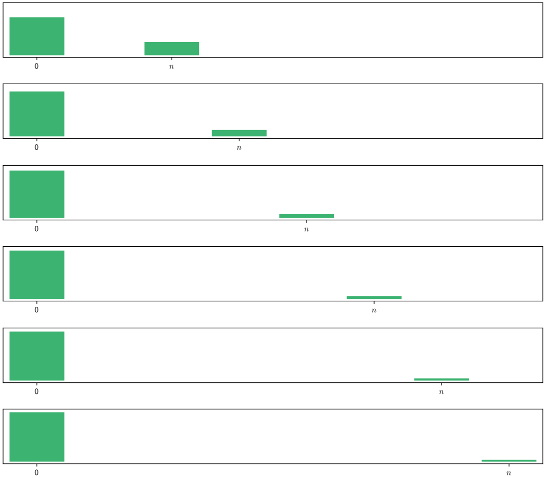

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Without bounds on the support, $\mathscr{P}_2(\mathbb{R}^d)$ is not locally compact.

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Using directional derivatives

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The choice of test functions

Define $\mathcal{T}_{\pm} := \{(t,\mu) \mapsto \psi(t) \pm \varphi(\mu) \mid \psi \in \mathcal{C}^1([0,T],\mathbb{R}),\ \varphi$ locally Lip and semiconcave$\}$.

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▷ Issue 1 solved! ◁

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Step 3. implies to minimize a l.s.c function, but no local compactness.

The smooth Ekeland principle

Let (X, d) be a complete metric space.

Gauge-type functions Any lower semicontinuous $\rho : X \times X \mapsto [0, \infty]$ satisfying $\rho(x, x) = 0$ for all $x \in X$, and $\forall \varepsilon > 0$, $\exists \eta > 0$ such that $\rho(x, y) \leqslant \eta$ implies $d(x, y) \leqslant \varepsilon$.

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Theorem – Borwein-Preiss [BP87] Let $f : X \mapsto \mathbb{R} \cup \{\infty\}$ be proper, lsc and lower bounded. Let ρ be gauge-type, $(\delta_i)_i \subset \mathbb{R}_*^+$, and $x_0 \in X$ such that $f(x_0) \leqslant \inf_X f + \varepsilon$.

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Gauge-type functions Any lower semicontinuous $\rho : X \times X \mapsto [0,\infty]$ satisfying $\rho(x,x) = 0$ for all $x \in X$, and $\forall \varepsilon > 0$, $\exists \eta > 0$ such that $\rho(x,y) \leqslant \eta$ implies $d(x,y) \leqslant \varepsilon$.

Theorem – Borwein-Preiss [BP87] Let $f : X \mapsto \mathbb{R} \cup \{\infty\}$ be proper, lsc and lower bounded. Let ρ be gauge-type, $(\delta_i)_i \subset \mathbb{R}_*^+$, and $x_0 \in X$ such that $f(x_0) \leqslant \inf_X f + \varepsilon$. Then there exist $y \in X$ and a sequence $(x_i)_{i=0}^{\infty} \subset X$ such that

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Illustration of Borwein-Preiss

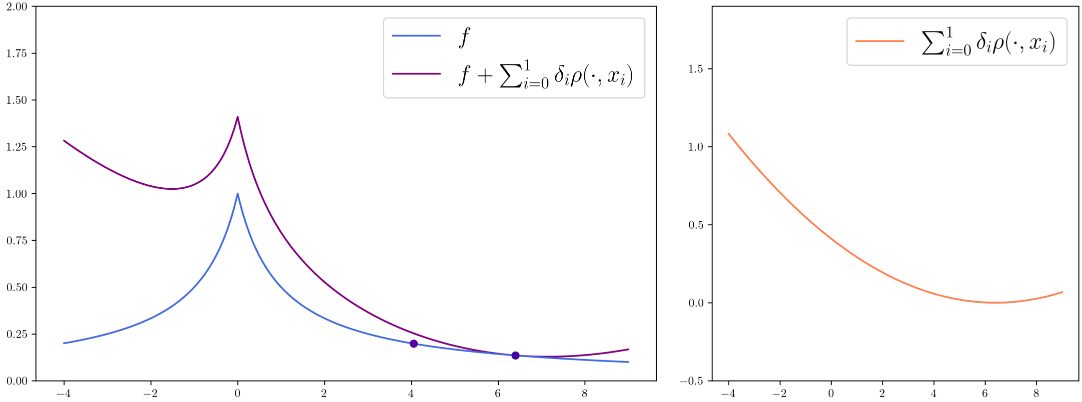

Figure: Iterative construction with $f(x) = (1 + |x|)^{-1}$, $\delta_i = 0.01/(1+i)^2$, $\rho(x, y) = |x - y|^2$.

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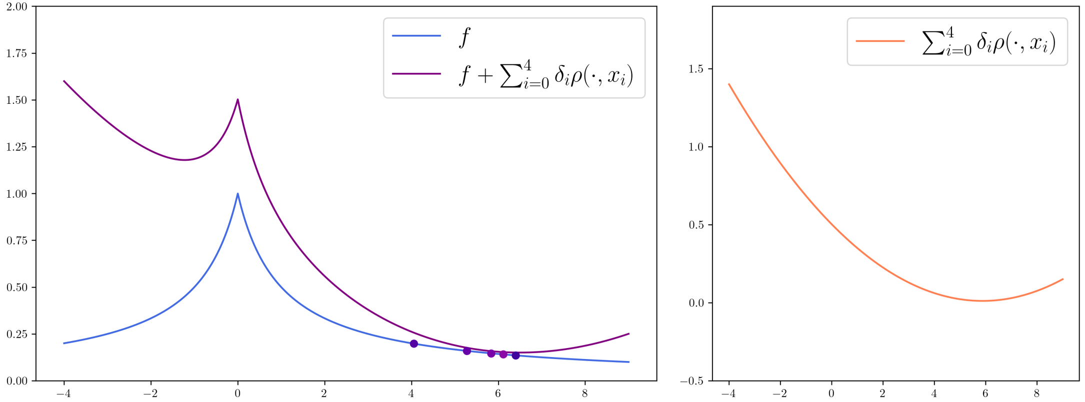

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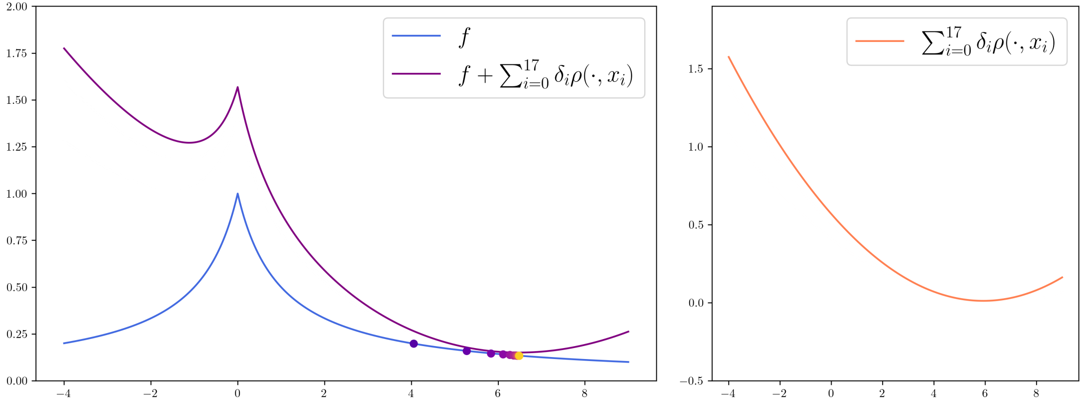

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- Apply the definition of viscosity, some estimate machinery to get rid of the perturbation.

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    where $C>0$ is a constant related to the speed of the propagation of information.
- The "swallowing trick" is a simple idea, but requires a large enough test function set.

Table of Contents

What is done

Theorem – Comparison principle ([JPZ23]) Assume $f : \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d) \times U \mapsto T\mathbb{R}^d$ is Lip. and bounded. Let v, w be Lipschitz and bounded sub/supersolutions of (HJ). Then

$$\inf_{(t,\mu)\in[0,T]\times\mathscr{P}_2(\mathbb{R}^d)} [w(t,\mu) - v(t,\mu)] \geqslant \inf_{\mu\in\mathscr{P}_2(\mathbb{R}^d)} [w(T,\mu) - v(T,\mu)] \, .$$

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Theorem Assume $\mathcal{J} : \mathscr{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}$ and f are Lip. and bounded. The value function V is the unique Lipschitz and bounded viscosity solution of (HJ).

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- using Measure Differential Equations ([Pic19, CCDMP21])

Thank you!

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