Using Optimal Transport to define viscosity solutions of control problems

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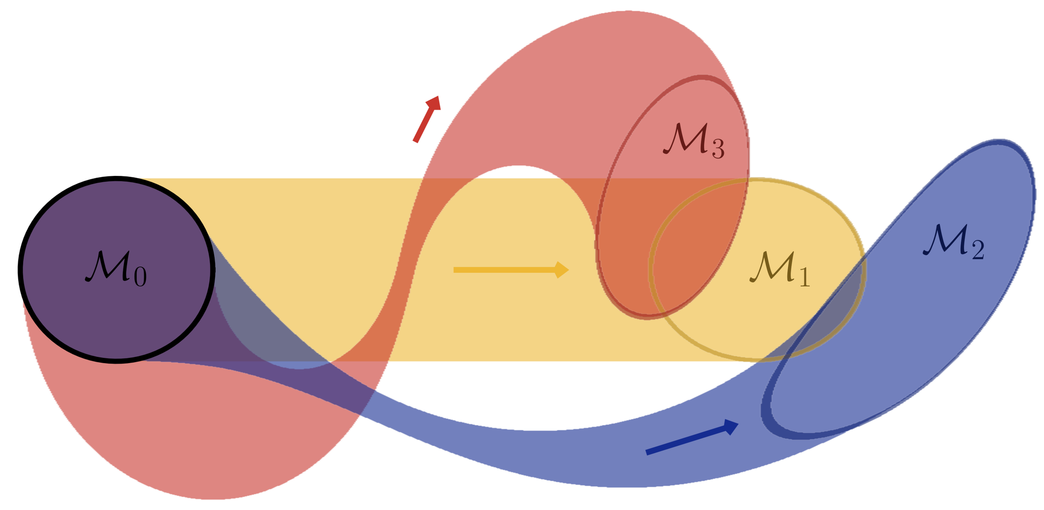
Motivation — Control problems on measures

We settle ourselves in the space of probability measures $\mathcal{P}_2(\mathbb{R}^d)$, endowed with the Monge-Kantorovitch distance $d_{\mathcal{W}}$ with $p = 2$. In $\mathcal{P}_2(\mathbb{R}^d)$, absolutely continuous curves are characterized as the solutions of the continuity equation [AGS05]. We consider time-dependant measurable controls $u(\cdot)$ valued in a compact $U \subset \mathbb{R}^\kappa$. Finding the trajectory minimizing some cost $\mathfrak{J}$ leads to study

$$V(t,\nu) \coloneqq \inf \left\{ \mathfrak{J}\left(\mu_T^{t,\nu,u}\right) \;\middle|\; (\mu_s^{t,\nu,u})_{s\in[t,T]} \text{ is a solution of (CE) with } \mu_t = \nu \text{ and control } u(\cdot) \right\}. \quad (1)$$

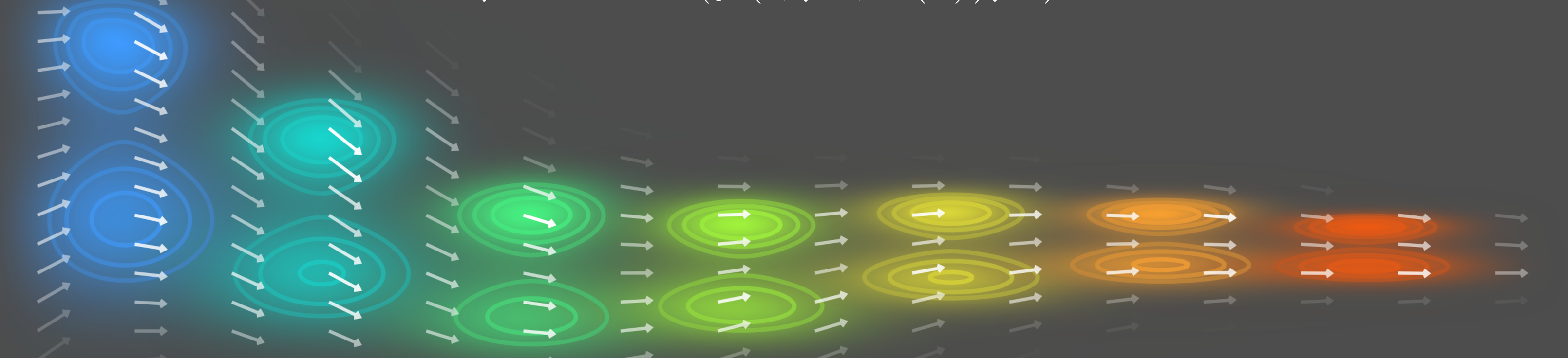


Such models find burgeoning applications in physics, biology, image processing or crowd simulation.

Characteristics in Wasserstein

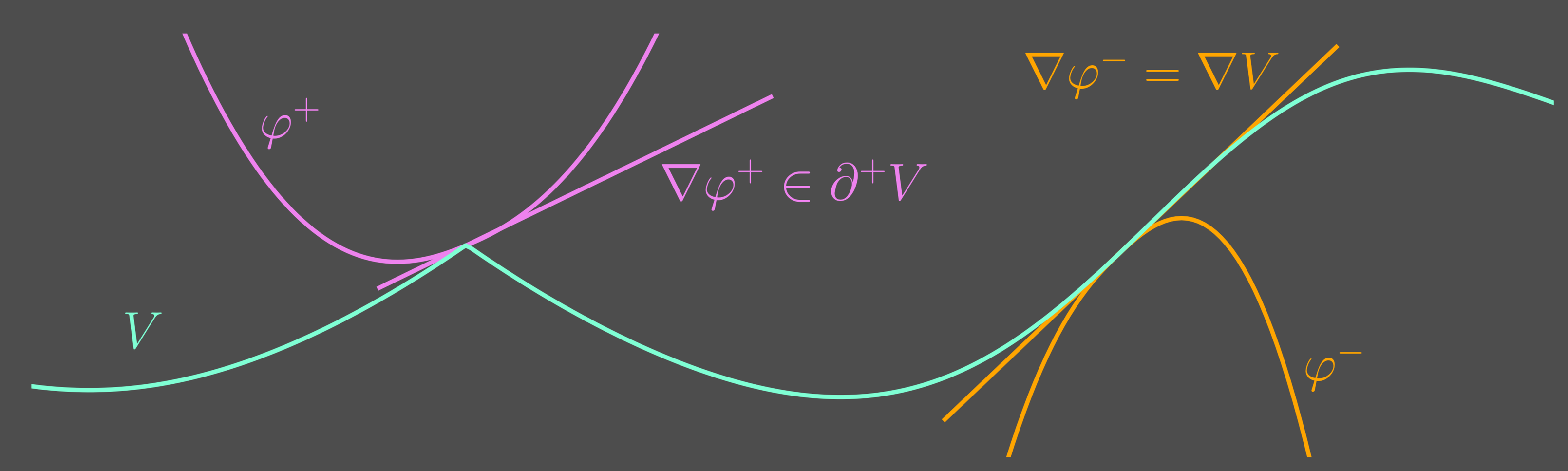
The controlled continuity equation (CE) is defined in distribution sense, and well-posed for our Lip. and bounded $f : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times U \mapsto T\mathbb{R}^d$.

$$\partial_s \mu_s + \operatorname{div}\left(f(\cdot, \mu_s, u(s))\mu_s\right) = 0 \quad \text{(CE)}$$



Viscosity for control problems

In Euclidian spaces, differential equations arising in control problems are understood using either test functions or semidifferentials.



Differential tools — Derivation along the geodesics

In the Euclidian space, one follows a geodesic by moving along a given vector. An analogy stands over measures, provided one accepts that *tangent directions* at μ will be represented as a special subset $\mathcal{P}(T\mathbb{R}^d)_{\mu,o}$ of the probabilities over the underlying tangent space, in bijection with the optimal transport plans. The completion of such a set gives a *tangent cone* (see Def 1). The exponential map $t \mapsto \exp_\mu(t\cdot\xi) \coloneqq (\pi_x + t\pi_v)\#\xi$ then replaces linear displacements in directional derivatives (see Def 2). Instead of a gradient, it is the map $\xi \mapsto D_\mu g(\mu)(\xi)$ of directional derivatives that will be used to define the Hamiltonian. Any locally Lipschitz and DC (difference of semiconvex) function admits such a differential, and fortunately, the squared Wasserstein distance happens to be so!

Def 1 — Tangent cone [Gig08]

Denote $\mathcal{P}_2(TE)_\mu \ni \xi \sim_\mu \bar{\xi}$ if $W_\mu(\xi,\bar{\xi}) = 0$, with

$$W_\mu(\xi,\bar\xi) \coloneqq \varlimsup_{t\searrow 0} \frac{d_{\mathcal{W}}(\exp_\mu(t\cdot\xi), \exp_\mu(t\cdot\bar\xi))}{t}.$$

The tangent cone to $\mathcal{P}_2(\mathbb{R}^d)$ at μ is defined as

$$T_\mu\mathcal{P}_2(E) \coloneqq \overline{\mathcal{P}_2(TE)_{\mu,o}/\sim_\mu}^{W_\mu}.$$

Def 2 — Derivative along geodesics

Given $g : \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}$ and $\xi \in \mathcal{P}_2(TE)_{\mu,o}$, let

$$D_\mu g(\mu)(\xi) \coloneqq \lim_{t\searrow 0} \frac{g(\exp_\mu(t\cdot\xi)) - g(\mu)}{t}.$$

If g is Lipschitz and DC, the limit exists, and $D_\mu g(\mu)$ extends to a Lipschitz and positively homogeneous function over $(T_\mu\mathcal{P}_2(E), W_\mu)$.

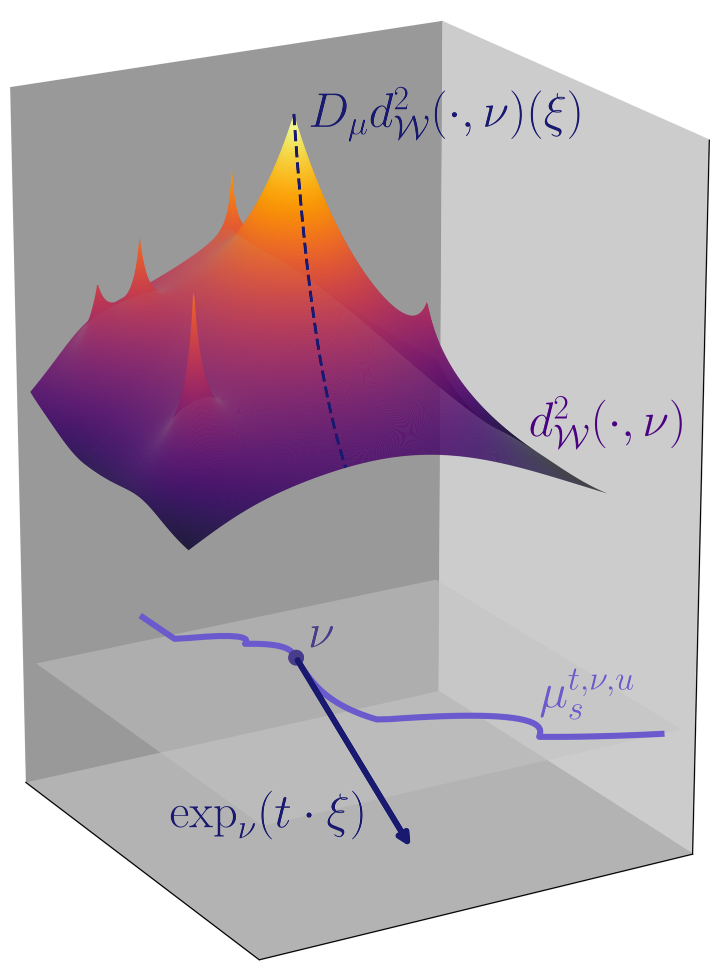
Def 3 — Metric cotangent bundle

Let $\mathcal{C}_\mu$ be the set of W_μ–Lipschitz and positively homogeneous maps from $T_\mu\mathcal{P}_2(E)$ to $\mathbb{R}$. The *metric cotangent bundle* is

$$\mathbb{T}(\mathcal{P}_2(\mathbb{R}^d)) \coloneqq \bigcup_{\mu\in\mathcal{P}_2(\mathbb{R}^d)} \{\mu\} \times \mathcal{C}_\mu. \quad \text{(MCB)}$$

$\mathbb{T}$ is used as a metric analogue of the dual space.

Characterization of the solution — Hamilton-Jacobi



Let the Hamiltonian $H : \mathbb{T} \mapsto \mathbb{R}$ be given by $H(\mu,p) \coloneqq \sup_{u\in U} -p\left(\pi^\mu \circ f(\cdot,\mu,u)\#\mu\right)$, where $\mathbb{T}$ comes from Def 3, and

$$\mathfrak{T}_\pm \coloneqq \left\{ (t,\mu) \mapsto \psi(t) \pm \sum_{i\in\mathbb{N}} \delta_i d^2_{\mathcal{W}}(\mu,\sigma_i) \;\middle|\; \psi \in \mathcal{C}^1([0,T]),\ \delta_i \geqslant 0,\ (\delta_i)_i \in l^1,\ \sigma_i \in \mathcal{P}_2(\mathbb{R}^d), \text{ and } \operatorname{diam}\{\sigma_i\} < \infty \right\}.$$

By construction, the differential $D_\mu\varphi(t,\mu)$ exists and belongs to $\mathcal{C}_\mu$ for all $\varphi \in \mathfrak{T}_\pm$ (see Def 3). We consider the equation

$$-\partial_t V(t,\mu) + H(\mu, D_\mu V(t,\mu)) = 0, \qquad V(T,\mu) = \mathfrak{J}(\mu). \quad \text{(HJ)}$$

Def 4 *A function $u : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}$ is a sub/supersolution of (HJ) if it is usc/lsc, and if for any test function $\varphi \in \mathfrak{T}_\pm$ such that $u - \varphi$ attains a max/minimum in $(t,\mu) \in [0,T[\times\mathcal{P}_2(\mathbb{R}^d)$, there holds $\pm[-\partial_t\varphi(t,\mu) + H(\mu, D_\mu\varphi(t,\mu))] \leqslant 0$. It is a solution of (HJ) if it is both a sub and supersolution, and satisfies the terminal condition.*

Def 4 allows for a weak comparison principle, relying on arguments inspired from the semidifferential notion of [JMQ22].

Theorem — Comparison principle Assume f is Lipschitz and bounded, and let v, w be Lipschitz and bounded sub/supersolution of (HJ). Then $\sup_{(t,\mu)\in[0,T]\times\mathcal{P}_2(\mathbb{R}^d)}(v(t,\mu) - w(t,\mu)) \leqslant \sup_{(t,\mu)\in\{T\}\times\mathcal{P}_2(\mathbb{R}^d)}(v(t,\mu) - w(t,\mu))$.

Solution of (HJ) — [JPZ23]

Assume that the dynamic f and the terminal cost $\mathfrak{J}$ are Lipschitz-continuous and bounded, and that the function set $\{f(\cdot,\mu,u) \mid u \in U\} \subset \mathcal{C}(\mathbb{R}^d, T\mathbb{R}^d)$ is convex for all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. Then the value function V, defined in (1), is the unique Lipschitz and bounded solution of (HJ) in the sense of Def 4.

[AGS05] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. *Gradient Flows.* Lectures in Mathematics ETH Zürich. Birkhäuser-Verlag, Basel, 2005.
[Gig08] Nicola Gigli. *On the Geometry of the Space of Probability Measures Endowed with the Quadratic Optimal Transport Distance.* PhD thesis, Scuola Normale Superiore di Pisa, Pisa, 2008.
[JMQ22] Chloé Jimenez, Antonio Marigonda, and Marc Quincampoix. Dynamical systems and Hamilton-Jacobi-Bellman equations on the Wasserstein space and their L2 representations. 2022.
[JPZ23] Othmane Jerhaoui, Averil Prost, and Hasnaa Zidani. Viscosity solutions of centralized control problems in measure spaces, 2023.

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